On the Origin of the Korteweg-de Vries Equation

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Abstract

The Korteweg-de Vries equation has a central place in a model for waves on shallow water and it is an example of the propagation of weakly dispersive and weakly nonlinear waves. Its history spans a period of about sixty years, starting with experiments of Scott Russell in 1834, followed by theoretical investigations of, among others, Lord Rayleigh and Boussinesq in 1871 and, finally, Korteweg and De Vries in 1895.

In this essay we compare the work of Boussinesq and Korteweg-de Vries, stressing essential differences and some interesting connections. Although there exist a number of articles, reviewing the origin and birth of the Korteweg-de Vries equations, connections and differences, not generally known, are reported.

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1 Introduction

It was in the *"interest of Higher Truth"* that professor Martin Kruskal, at the conference in commemoration of the centennial of the publication of the Korteweg-de Vries paper in the Philosophical Magazine [1], claimed that *"he, together with professor Norman Zabusky, was the person, who more than anyone else, resuscitated the Korteweg-de Vries equation after its long period of, if not oblivion, at least neglect"*, [2]. Indeed it is well-known that in the follow-up of their 1965 paper in the Physical Review Letters [3], "Interactions of Solitons in a collisionless plasma and the recurrence of initial states", a real explosion of research on this and related equations appeared in the journals. Many developments in several fields of pure and applied mathematics, physics, chemistry, biology and engineering followed. Restricting to mathematics we mention analysis, integrability of nonlinear systems, Lie-algebra's, differential geometry, quantum and statistical mechanics, [2, 4, 5].

In this essay we direct our attention to the origin of the Korteweg-de Vries equation and its birth which has been a long process and spanned a period of about sixty years, beginning with the experiments of Scott-Russell in 1834 [6], the investigations of Boussinesq and Rayleigh around 1870 [7–11] and finally ending with the article by Korteweg and De Vries in 1895 [1].

In simplified form the Korteweg-de Vries equation reads

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{1.1}$$

and it is the result of research concerning long waves in shallow water; x and t denote position and time and $u = u(x,t)$ the wave surface.

Nowadays, it is hard to understand for mathematicians not specialized in fluid mechanics, that a subject as this could raise such a wide spread interest. However, this was not the case in the nineteenth century when the study of water waves was of vital interest for applications in naval architecture and for the knowledge of tides and floods. Notably in England and France much research was spent on the study of water waves of several kinds, in England by, among others, Scott–Russell, Airy, Stokes, McCowan, Lord Rayleigh and Lamb and in France by Lagrange, Clapeyron, Bazin, St. Venant and Boussinesq.

In some treatises and textbooks on "soliton theory" a short survey of the early history is presented [12, 13], but apparently it has not been the intention of the authors to dwell extensively upon the considerations and the mathematical analysis of those present at the cradle of the equation that became later known as the Korteweg-de Vries equation. Nevertheless, there are some review papers where more specific attention has been given to investigations related to the Korteweg-de Vries equation. We mention in particular the reviews by Bullough [14], Bullough and Caudry [15], Miles [16] and the recent impressive extensive article by Darrigol [17] and a letter in the Notices of the A.M.S. by Pego [18].

In these articles the work by Boussinesq on the one side and that of Korteweg and De Vries on the other side have been discussed. Studying these papers, the present author became aware of some inaccuracies regarding the relevance

and the significance of the work by Korteweg and De Vries, maybe even a slight animosity over the priority of the discovery of the equation. For example, Miles, Darrigol and Pego suggest that Korteweg and De Vries were presumably unaware of the work by Boussinesq. This is to be doubted because in their article reference has been made to the Comptes Rendus papers by Boussinesq [7] and St. Venant [19]. Besides this, the historian B. Willink [20] has presented the author with a handwritten copy of De Vries, containing an excerpt of the paper by St. Venant [19] and there appears a clear reference to the *"Essai sur la théorie des eaux courantes"* [10], which proves that De Vries was certainly aware of the existence of Boussinesq's research. Pego writes *"It is not clear why Korteweg and De Vries thought the permanence of the solitary wave still controversial in 1895"* [18]. This is in contrast with the introduction of the KdV article, where it is stated *"They (Lord Rayleigh and McCowan) are as it seems to us, inclined to the opinion that the wave is only stationary to a certain approximation. It is the desire to settle this question definitively which has led us into somewhat tedious calculations, which are to be found at the end of our paper"* , [1].

It is evident that Korteweg and De Vries, wanting to check the theory of long waves in shallow water, use an independent approach. It is our intention to illustrate this in the next sections, pointing out not only differences but also close connections in both theories. Here we already give some examples. Boussinesq used a fixed coordinate system and Korteweg and De Vries a coordinate system moving with the wave. The central equations in Boussinesq's analysis are the continuity equation and an expression for the wave velocity [9], whereas the Korteweg-de Vries equation is the central equation to which Korteweg and De Vries frequently revert in the course of their paper [1]. A simple substitution of the wave velocity into the continuity equation yields immediately the Korteweg-de Vries equation in its full glory. However, Boussinesq did not do this, otherwise it may well be that the history of the long stationary wave had taken a different course. Pego [18] pointed out that the Korteweg-de Vries equation appeared already in a footnote on page 360 of Boussinesq's 680 pages vast volume *"Mémoir sur la théorie des eaux courantes"* [10], that appeared in 1877, well before the publication of the Korteweg-de Vries paper in 1895. However, this footnote on the Korteweg-de Vries equation and also Boussinesq's expression for the wave velocity are only valid when the wave vanishes at infinity, while this is not necessary in the theory of Korteweg and De Vries. Therefore, Boussinesq uses another approach for treating *steady periodic* waves than Korteweg and De Vries, who presented a unified treatment for steady waves, not only vanishing at infinity but for waves being periodic as well. It is not only the equation , but also its applicability that is important. It seems that this is not always sufficiently realized or even mentioned in the literature. Darrigol spends in his essay only one page to the Korteweg-de Vries equation under the heading *"The so-called Korteweg-de Vries equation"* [17]. It is only a whim of Tyche, the daughter of Zeus and the personification of fate, that Zabusky and Kruskal attributed the names of Korteweg and De Vries to our equation and not that of Boussinesq, who merits of course the token of priority.

In the following account we present a review of the work by Boussinesq and Korteweg and De Vries; as to Boussinesq, most of our attention is directed to his long article in the Journal des Mathematiques Pures et Appliquées [9], which is more accessible than his vast memoir [10]. The author, not a historian, is well aware that he may have overlooked or deleted important facts, but nevertheless he hopes that this study may disclose some generally unknown aspects of the early history of the Korteweg-de Vries equation.

2 Scott Russell's experiments

The story of the discovery of the "Wave of Translation" by John Scott Russell in 1834, has been recorded in many books concerning "Soliton" theory, the more so because Scott Russell's account is fascinating and even full of emotion, hardly expected in a scientific paper. Therefore, our account will be rather short and the interested reader is referred to his "Report on Waves" [6, 4, 12, 14]. It was in the year of 1834 that the Scottish naval architect followed on horseback a towboat, pulled by a pair of horses along the Union Canal, connecting Edinburgh and Glasgow. However, the boat was suddenly stopped in its speed - presumably by some obstacle - but not the mass of water, which it had put in motion. Our engineer perceived a very peculiar phenomenon: a nice round and smooth wave - a well defined heap of water - loosened itself from the stern and moved off in forward direction without changing its form with a speed of about eight miles an hour and about thirty feet long and one or two feet in height. He followed the wave on his horse and after a chase of one or two miles he lost the heap of water in the windings of the channel [6]. Many a physicist would not be inclined to analyze this phenomenon and leave it as it is, not so Scott Russell discovering something very peculiar in a seemingly ordinary event. He designed experiments generating long waves in long shallow basins filled with a layer of water and he investigated the phenomenon he had observed. He studied the form of the waves, their speed of propagation and stability, clearly perceptible in progressing positive waves, but not in progressing negative waves. A schematic view of these experiments is shown in figure 1, which is adapted from Remoissenet [21].

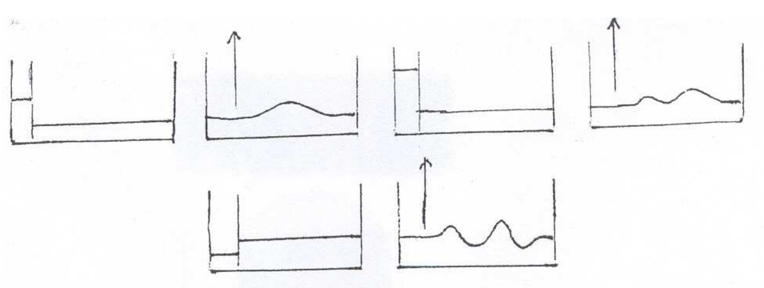

Figure 1: Scott Russell's experiments

For an extensive historical study the reader may consult the papers by Bullough [14] and Darrigol [17]. As mentioned in the introduction, there existed in England and France a rich tradition in the mathematical description of hydrodynamic phenomena such as wave motions in fluids. Scott Russell challenged the mathematical community to prove theoretically the existence of his solitary wave and "to give an a priori demonstration a posteriori" i.e. to show the possible existence of a stable solitary wave propagating without change of form.

It is not unusual that new discoveries or new ideas encounter resistance from established convictions. We take from Rayleigh's paper "On Waves" in the Philosophical Magazine [11, pp 257-279, 1876] the following quotations. Airy, an authority on the subject, writes in his treatise on "Tides and Waves" [22]: *"We are not disposed to recognize this wave (discovered by Scott Russell) as deserving the epithets "great" or "primary", and we conceive that ever since it was known that the theory of shallow waves of great length was contained in the equation* $\frac{\partial^2 X}{\partial t^2} = g\kappa \frac{\partial^2 X}{\partial x^2}$ *the theory of the solitary wave has been perfectly well known".* Further *"Some experiments were made by Mr. Russell on what he calls a negative wave. But (we know not why) he appears not to have been satisfied with these experiments and had omitted them in his abstract. All of the theorems of our IVth section, without exception, apply to these as well as to positive waves, the sign of the coefficient only being changed".* Probably it was also Airy who expressed for the first time as his opinion that long waves in a canal with rectangular cross section must necessarily change their form as they advance, becoming steeper in front and less steep behind and in this he was supported by Lamb and Busset [1, 22, 23]. Stokes believed that the only permanent wave should be basically sinusoidal, but later on he admitted that he had made a mistake, see also our section 7.

On the other hand he writes [24]: *"It is the opinion of Mr. Russell that the solitary wave is a phenomenon "sui generis", in no wise deriving its character from the circumstance of the generation of the wave. His experiments seem to render this conclusion probable. Should it be correct, the analytical character of the solitary wave remains to be discussed".*

The *"a priori demonstration a posteriori"* asked for by Scott Russell was finally given, first by J. Boussinesq in 1871 [7-10], some time later in 1876, by Lord Rayleigh [11] and in order to remove still existing doubts over the existence of the solitary wave by G. de Vries [25] and by D.J. Korteweg and G. de Vries in 1895 [1].

In the next section we present first a concise account of the contribution by Rayleigh, since it is short and leads directly to the heart of the matter. Moreover, this paper has been of great influence on the research of Korteweg and De Vries. Consecutively, we discuss in the other sections the investigations of Boussinesq and Korteweg-de Vries and we finish with some concluding remarks.

3 Rayleigh's Solution

Be given an incompressible irrotational flow in a canal with a constant rectangular cross-section, fig. 2.

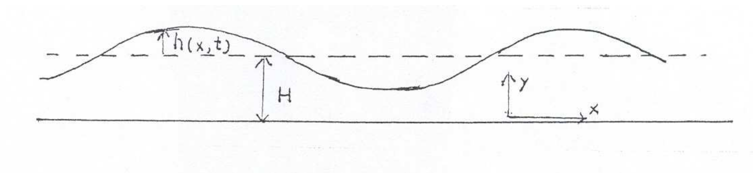

Figure 2: Wavesurface

The coordinates of a fluid particle are given by the coordinates x and y, the undisturbed depth of the canal by the constant H and the wave surface by $H + h(x,t)$. Another essential assumption is that the wave length is large in comparison with H.

As has been mentioned in the preceding section, Airy had already remarked that the theory of shallow waves of great length is contained in the equation

$$\frac{\partial^2 X}{\partial t^2} = g\kappa \frac{\partial^2 X}{\partial x^2}$$

(where $\kappa = H$ and g the constant of gravity). The wave velocity is $\sqrt{g\kappa}$, a result generally known since Lagrange in 1786. However, this result is only valid as a first order approximation, where h/H may be neglected.

Rayleigh remarked that for this value of the wave velocity the so-called free surface condition (equilibrium of pressure) is only satisfied whenever the ratio h/H may be neglected, but if this is not the case it is impossible to have a wave in still water with velocity $\sqrt{g\kappa}$ and at the same time propagating without change of form. In order to cure this discrepancy with Scott Russell's experimental results, he proposes to look for a more accurate approximation of the wave velocity ([11], pp 252-253). Rayleigh assumes the existence of a stationary wave, vanishing at infinity, and by adding to the fluid a yet unknown constant basic velocity equal and opposite to that of the wave, he may omit the dependence on time. Since the flow is free of rotation, and incompressible, there exist a velocity potential ϕ and a stream function ψ, both satisfying Laplace's equation. The horizontal and vertical velocity components are given by

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

and a series expansion gives

$$u = \frac{\partial \phi}{\partial x} = f(x) - \frac{y^2}{2!} f''(x) + \frac{y^4}{4!} f^{(4)}(x) - \ldots \tag{3.1}$$

$$v = \frac{\partial \phi}{\partial y} = -y f'(x) + \frac{y^3}{3!} f^{(3)}(x) - \ldots \tag{3.2}$$

This expansion is justified because $f(x)$ is, due to the large wave lengths, a slowly varying function of x.

Integration yields the stream function

$$\psi = yf(x) - \frac{y^3}{3!}f''(x) + \frac{y^5}{5!}f^{(4)}(x) \tag{3.3}$$

constant along stream lines and hence also along the wave surface $y = H + h(x)$.

Let p be the pressure just below the wave surface, then we have the relation

$$-2\frac{p-C}{\rho} = 2g(H+h) + u_s^2 + v_s^2 := \tilde{\omega} \tag{3.4}$$

where ρ is the density, g the constant of gravity and C an integration constant; the suffix s denotes that the values of u and v are taken at the surface of the wave.

To satisfy in higher approximation the free surface condition, — p constant —, Rayleigh investigates how far it is possible to make $\tilde{\omega}$ constant by varying $h(x)$ as function of x. Using $u_s^2 + v_s^2 = u_s^2\{1 + (\frac{dh}{dx})^2\}$ and eliminating the unknown function f with the aid of (3.1) and (3.3) he obtains after a tedious calculation a differential equation for the wave form $y = H + h(x)$, viz.

$$\tfrac{1}{3}\left(\frac{dy}{dx}\right)^2 = 1 + Cy + \frac{u_0^2 + 2gH}{u_0^2H^2}y^2 - \frac{g}{u_0^2H^2}y^3 \tag{3.5}$$

where u_0 is the still unknown constant basic velocity that has been added to the flow and C is again an integration constant, ([11], pp 258-259). The cubic expression at the right hand side vanishes for $y = H$ with $x = \infty$ and for $y = H + h_0$ with h_0 the crest of the wave.

Elimination of C yields

$$u_0 = \sqrt{g(H+h_0)} \approx \sqrt{gH} + \tfrac{1}{2}h_0\sqrt{\frac{g}{H}} \tag{3.6}$$

which is also the wave velocity, and the equation (3.5) reduces to

$$\left(\frac{dh}{dx}\right)^2 + \frac{3}{H^3}h^2(h-h_0) = 0$$

with the solution

$$h(x) = h_0\,\mathrm{sech}^2\left(\sqrt{\frac{3h_0}{4H^3}}x\right). \tag{3.7}$$

This formula represents the "heap of water" (the "Great Wave") with the right wave velocity (3.6), as experimentally observed by Scott Russell and so he was finally vindicated after fourty years of much discussion in England.

Rayleigh finishes his article with the remark: *"I have lately seen a memoir by M. Boussinesq, Comptes Rendus, Vol. LXXII, in which is contained a theory of the solitary wave very similar to that of this paper. So as far as our results are common, the credit of priority belongs of course to M. Boussinesq".*

4 The Equations of Boussinesq

From Miles we learn in his essay [16] that Boussinesq (1842-1929) received his doctorate from the Faculté des Sciences, Paris, in 1867, occupied chairs at Lille from 1873 to 1885, and at the Sorbonne from 1885 to 1896. He made significant contributions to hydrodynamics and the theories of elasticity, light and heat. He wrote several papers on nonlinear dispersive waves [7, 8, 9] and a voluminous "mémoire", entitled "Essai sur la théorie des eaux courantes", presented to the Académie des Sciences in 1877, Vol. XXIII ([10], pp 1-680).

The most accessible publication is his article in the Journal de Mathématiques Pures et Appliquées in 1872 [9]. It has the verbose title *"Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide continu dans ce canal des vitesses sensiblement pareilles de la surface au fond"*. This paper subsumes the short Comptes Rendus [7, 8], whereas the monograph [10] gives also much information less relevant for this exposition concerning the "Great Wave".

He considers, in the same way as Rayleigh, long waves in a shallow canal with rectangular cross section; the fluid is supposed incompressible and rotation free, while friction, also along the boundaries of the canal, is neglected. Distinct from Rayleigh's article, Boussinesq introduces also a time variable, essential for the description of a dynamic phenomenon, and the coordinates of a fluid particle at time t are denoted by $(x, y) = (x(t), y(t))$, see fig. 2.

Be p the pressure in the fluid, ρ its density and (u, v) the velocity vector. The height of the water in equilibrium is again denoted by the constant H and the wave surface by the function $y = H + h(x, t)$. The wave length is supposed to be large and the amplitude h of the wave small in comparison with H and vanishing for $x \to \pm\infty$.

Boussinesq's exposition starts along the same lines as in the theory of Rayleigh. The main ingredients are a series development into powers of y, similar as in (3.1) – (3.3):

$$\phi = f - \frac{y^2}{2!} f'' + \frac{y^4}{4!} f^{(4)} - \dots \tag{4.1}$$

with the as yet unknown function $f = f(x, t)$ and valid for $0 < y < H + h(x, t)$.

Further, he uses the free surface condition

$$gh + \tfrac{1}{2}(u_s^2 + v_s^2) + \frac{\partial \phi_s}{\partial t} + \chi(t) = 0 \tag{4.2}$$

where $\chi(t)$ is an arbitrary function and where the suffix s refers to the wave surface. Under the assumption that the potential ϕ and its derivatives with respect to x, y and t vanish for $x \to \pm\infty$, the function $\chi(t)$ may be omitted. We shall see that this assumption is very essential in Boussinesq's theory and it is used again and again in his paper, see also section 7.

A second boundary condition follows from the kinematic equation

$$v_s = \frac{dh}{dt} = \frac{\partial h}{\partial t} + u_s \frac{\partial h}{\partial x} \tag{4.3}$$

Substitution of the series expansions of the potential and the velocity components into (4.2) and (4.3) results into two equations, containing $h(x,t)$ and $f(x,t)$. Elimination of $f(x,t)$ gives in a first approximation, with $h(x,t)$ small in comparison with H, the wave equation of Lagrange:

$$\frac{\partial^2 h}{\partial t^2} = gH\frac{\partial^2 h}{\partial x^2}$$

A second higher approximation yields the well known equation of Boussinesq

$$\frac{\partial^2 h}{\partial t^2} = gH\frac{\partial^2 h}{\partial x^2} + gH\frac{\partial^2}{\partial x^2}\left[\frac{3h^2}{2H} + \frac{H^2}{3}\frac{\partial^2 h}{\partial x^2}\right] \tag{4.4}$$

This equation may be simplified by restricting the theory to waves propagating into only one direction, say the positive x-axis. Denoting the wave velocity by $\omega(x,t)$ and using the conservation of mass

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(\omega h) = \frac{\partial h}{\partial t} + \omega\frac{\partial h}{\partial x} + h\frac{\partial \omega}{\partial x} = 0 \tag{4.5}$$

we obtain after substitution into (4.4) and integration with respect to x

$$\frac{\partial}{\partial t}(\omega h) + gH\frac{\partial}{\partial x}(h + \frac{3}{2}\frac{h^2}{H} + \frac{H^2}{3}\frac{\partial^2 h}{\partial x^2}) = 0 \tag{4.6}$$

To make progress it is desirable to have an explicit expression for $\omega(x,t)$, because substitution into (4.5) or (4.6) gives a differential equation for the wave surface $h(x,t)$. To this end Boussinesq introduces without a clear motivation the function

$$\psi(x,t) = h\cdot(\omega - \sqrt{gH}) - \frac{\sqrt{gH}}{2}\left(\frac{3}{2}\frac{h^2}{H} + \frac{H^2}{3}\frac{\partial^2 h}{\partial x^2}\right). \tag{4.7}$$

Differentiation of this expression with respect to t and replacing $\frac{\partial}{\partial t}$ by $-\sqrt{gH}\frac{\partial}{\partial x}$, which does not disturb the order of approximation in the second term, yields

$$\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial t}(\omega h) - \sqrt{gH}\frac{\partial h}{\partial t} + \frac{gH}{2}\frac{\partial}{\partial x}\left(\frac{3}{2}\frac{h^2}{H} + \frac{H^2}{3}\frac{\partial^2 h}{\partial x^2}\right)$$

Substitution of (4.6) and (4.5) gives $\frac{\partial \psi}{\partial t} = \sqrt{gH}\frac{\partial \psi}{\partial x}$, from which it follows that $\psi \equiv 0$ for a wave propagating in the positive x-direction. Hence, Boussinesq obtains from (4.7) the important result

$$\omega(x,t) = \sqrt{gH} + \sqrt{gH}\left(\frac{3h}{4H} + \frac{H^2}{6h}\frac{\partial^2 h}{\partial x^2}\right) \tag{4.8}$$

From (4.5) and (4.8) one may obtain the differential equations

$$h\frac{dh}{dt} = h\left(\frac{\partial h}{\partial t} + \omega\frac{\partial h}{\partial x}\right) = -h^2\frac{\partial \omega}{\partial x} = -\frac{\partial}{\partial x}(h^2\omega) + 2\omega h\frac{\partial h}{\partial x}$$

and

$$\frac{dh}{dt} = -\frac{1}{4}\sqrt{\frac{g}{H}}\frac{1}{h}\frac{\partial}{\partial x}\left[h^3\left\{1+\frac{2}{3}H^3\frac{1}{h}\left(\frac{\partial}{\partial x}\frac{1}{h}\frac{\partial h}{\partial x}\right)\right\}\right] \tag{4.9}$$

or after passing to the new variable $h\,dx = -d\sigma$

$$\frac{dh}{dt} = \frac{1}{4}\sqrt{\frac{g}{H}}\frac{\partial}{\partial \sigma}\left[h^3\left\{1+\frac{2}{3}H^3\frac{\partial^2 h}{\partial \sigma^2}\right\}\right]. \tag{4.10}$$

It appears from (4.8) that the wave velocity differs from point to point at the wave surface and so it is expected that the wave should change its form during its course, which is one of the main issues in pursuing Scott Russell's experiments. However, the wave will only be stationary whenever $\omega(x,t)$ is constant. It is appropriate to make here some comments:

1. The introduction of the function ψ in (4.7) is not a priori clear and motivated. With the aid of (4.5) we may write instead of (4.6)

$$\frac{\partial}{\partial t}\left\{h(\omega-\sqrt{gH})\right\} - \sqrt{gH}\frac{\partial}{\partial x}\left\{h(\omega-\sqrt{gH})\right\} + gH\frac{\partial}{\partial x}\left(\frac{3h^2}{2H}+\frac{H^2}{3}\frac{\partial^2 h}{\partial x^2}\right) = 0.$$

Replacing $\frac{\partial}{\partial t}$ by $-\sqrt{gH}\frac{\partial}{\partial x}$ and integrating with respect to x, we get the result (4.8); remember $h(x,t)$ and its derivatives vanish for $x \to \pm\infty$.

2. Another proof of (4.8) was given many years later in 1885 by St. Venant [19]. It may be that he was not satisfied with Boussinesq's derivation. He applied another approach, using the mean value of the horizontal component of the velocity vector

$$U(x,t) = \frac{1}{H+h(x,t)}\int_0^{H+h(x,t)} u\,dy$$

3. The equations (4.9) and (4.10) contain implicitly the wave velocity ω. This dependence on ω can simply be eliminated by the substitution of (4.8) into (4.5). If Boussinesq had carried out this small operation he had obtained the Korteweg-de Vries equation long "avant la lettre", viz.

$$\frac{\partial h}{\partial t} + \sqrt{\frac{g}{H}}\frac{3}{2}\frac{\partial}{\partial x}\left(\frac{2}{3}Hh+\frac{1}{2}h^2+\frac{H^3}{9}\frac{\partial^2 h}{\partial x^2}\right) = 0 \tag{4.11}$$

valid for waves vanishing at infinity. This equation does not differ essentially from the Korteweg-de Vries equation as presented in the Korteweg-de Vries paper in the Philosophical Magazine; the coordinates in (4.11) refer to a fixed (x,t) frame, whereas Korteweg and De Vries used a moving frame, see also next section.

4. As mentioned in the Introduction of this paper, R. Pego [18] and also O. Darrigol ([17], pp 47) have discovered in a footnote on page 360 of the "Essai sur la théorie des eaux courantes", that Boussinesq had found already in 1876 the Korteweg-de Vries equation, mind without recourse to the expression (4.8) for the wave velocity [10]. In fact he used instead of (4.7) the function

$$\psi_1(x,t) = \frac{\partial h}{\partial t} + \sqrt{gH}\frac{\partial h}{\partial x} + \frac{1}{2}\sqrt{gH}\frac{\partial}{\partial x}\left(\frac{3h^2}{2H}+\frac{H^2}{3}\frac{\partial^2 h}{\partial x^2}\right) = -\frac{\partial \psi}{\partial x}$$

and using $\psi \equiv 0$ he gets equation (4.11).

Consecutively, the wave velocity $\omega(x,t)$ may be determined by the integration of (4.5), i.e.

$$\omega(x,t) = \frac{1}{h(x,t)} \int_{-\infty}^{x} \left(-\frac{\partial h}{\partial t} \right) dx$$

5. The bidirectional Boussinesq equation (4.4) can be factorized as

$$\left(\frac{\partial}{\partial t} - \sqrt{gH} \frac{\partial}{\partial x} \right) \left\{ \frac{\partial h}{\partial t} + \sqrt{\frac{g}{H}} \frac{3}{2} \frac{\partial}{\partial x} \left(\frac{2}{3} H h + \frac{1}{2} h^2 + \frac{H^3}{9} \frac{\partial^2 h}{\partial x^2} \right) \right\} = 0 \quad (4.12)$$

from which it immediately follows that the unidirectional Korteweg-de Vries equation (4.11) is contained in Boussinesq's equation (4.4).

5 The Korteweg-de Vries Equation

We start with a few biographical data of Korteweg and De Vries [20]. Diederik Johannes Korteweg (1848-1941) received in 1878 his doctorate at the University of Amsterdam, after defending his thesis on the propagation of waves in elastic tubes. His supervisor was J.D. van der Waals, renowned for his equation of state and the continuity of the gas and fluid phases. Korteweg occupied the chair of mathematics, mechanics and astronomy at the University of Amsterdam from 1881 to 1918; he published several papers on mathematics, classical mechanics, fluid mechanics and thermodynamics. We mention in particular his investigations on the properties of "folded" surfaces in the neighbourhood of singular points, work related to that of Van der Waals [26]. Another scientific achievement is the edition of the "Oevres Complètes" of Christiaan Huygens and Korteweg was the principal leader of this project in the period 1911-1927.

He inspired many young mathematicians who wrote their thesis under his supervision, among others Gustav de Vries and the famous L.E.J. Brouwer. Korteweg had a great influence on academic life in the Netherlands as appears from his leadership in several academic institutions. The thesis of Gustav de Vries, entitled "Bijdrage tot de Kennis der Lange Golven" [25] was published in 1894 and the paper in the Philosophical Magazine of 1895 is an excerption of this thesis. De Vries published papers on cyclones in 1896 and 1897 and two papers "Calculus Rationum" in the proceedings of the Royal Dutch Academy of Arts and Sciences in 1912. He teached mathematics at a gymnasium in Alkmaar and at a secondary school in Haarlem.

The author received from the grandsons of De Vries a copy of the scientific correspondence between Korteweg and De Vries. From this we know that Korteweg advised De Vries to study Rayleigh's method of the series expansion which has been explained in section 3 of this paper. He also suggested to include capillarity and to investigate long periodic waves.

Korteweg and De Vries start their article with the time dependent Rayleigh

expansions

$$u(x,t) = f(x,t) - \frac{y^2}{2!} f''(x,t) + \frac{y^4}{4!} f^{(4)}(x,t) - \dots \tag{5.1}$$

$$v(x,t) = -y f'(x,t) + \frac{y^3}{3!} f^{(3)}(x,t) - \dots \tag{5.2}$$

The effect of the surface tension in the free surface condition amounts to an extra term in (4.2):

$$gh + \tfrac{1}{2}(u_s^2 + v_s^2) + \frac{\partial \phi_s}{\partial t} + \chi(t) = \frac{T}{\rho} \frac{\partial^2 y_s}{\partial x^2} \tag{5.3}$$

where T is the surface tension and $\chi(t)$ again the arbitrary function depending only on time.

This arbitrary function is eliminated by Boussinesq with the aid of the assumption that f and its derivations vanish for $x \to \pm\infty$. Korteweg and De Vries drop this crucial assumption and $\chi(t)$ is eliminated by simply differentiating (5.3) with respect to x. It is now already noted that *periodic* waves are not a priori excluded from further discussion, this in contrast to Boussinesq, who used a different approach in his discussion of periodic waves, see section 7.1. Differentiating (5.3) with respect to x, the free surface condition (5.3) becomes

$$g\frac{\partial h}{\partial x} + u_s \frac{\partial u_s}{\partial x} + v_s \frac{\partial v_s}{\partial x} + \frac{\partial^2 \phi_s}{\partial t \partial x} - \frac{T}{\rho}\frac{\partial^3 y_s}{\partial x^3} = 0 \tag{5.4}$$

Besides this we need again the kinematic condition (4.3)

$$v_s = \frac{\partial h}{\partial t} + u_s \frac{\partial h}{\partial x}. \tag{5.5}$$

Korteweg and De Vries put $y_s = H + h(x,t)$ and $f(x,t) = q_0 + \beta(x,t)$ with q_0 an as yet undetermined constant velocity. Substitution of (5.1) and (5.2) into (5.4) and (5.5) gives in a first order approximation for h small and for a wave, progressing in the positive x-direction, the expression

$$h = h(x - (q_0 + \sqrt{gH})t).$$

Adding to the flow a velocity $q_0 = -\sqrt{gH}$, we obtain the Lagrange steady wave solution with

$$\frac{\partial h}{\partial t} = 0, \qquad \frac{\partial \beta}{\partial t} = 0,$$

and

$$\frac{\partial \beta}{\partial x} = -\frac{q_0}{H}\frac{\partial h}{\partial x} = -\frac{g}{q_0}\frac{\partial h}{\partial x} \text{ or } \beta = -\frac{g}{q_0}(h+a),$$

where a is an undetermined constant.

The next approximation is obtained by

$$f(x,t) = q_0 - \frac{g}{q_0}(h(x,t) + \alpha + \gamma(x,t))$$

with γ small in comparison with h and a. Substitution into (5.4) and (5.5) yields two equations for h and γ and elimination of $\gamma(x,t)$ gives finally the Korteweg-de Vries equation as it appeared for the first time in the thesis of De Vries [25]:

$$\frac{\partial h}{\partial t} = \frac{3}{2}\frac{g}{q_0}\frac{\partial}{\partial x}\left(\frac{1}{2}h^2 + \frac{2}{3}\alpha h + \frac{1}{3}\sigma\frac{\partial^2 h}{\partial x^2}\right) \tag{5.6}$$

with $\sigma = \frac{1}{3}H^3 - \frac{TH}{\rho g}$.

The addition of the velocity $q_0 - \frac{g}{q_0}\alpha = -\sqrt{gH} + \sqrt{g/H}\alpha$ to the flow may also be obtained by a transformation of the fixed (x,y) coordinate system to the moving frame

$$\xi = x - (\sqrt{gH} - \sqrt{\frac{g}{H}}\alpha)t, \quad \tau = t. \tag{5.7}$$

Hence, in this moving frame and forgetting about the added velocity, we get

$$\frac{\partial h}{\partial \tau} + \frac{3}{2}\sqrt{\frac{g}{H}}\frac{\partial}{\partial \xi}\left(\frac{1}{2}h^2 + \frac{2}{3}\alpha h + \frac{1}{3}\sigma\frac{\partial^2 h}{\partial \xi^2}\right) = 0. \tag{5.8}$$

This equation with $T = 0$ is equivalent with Boussinesq's result (4.11) and may be obtained by substititution of (5.7) into (4.11).

6 The Solitary Great Wave

Because of the equivalence of the differential equations (4.11) and (5.8) for the surface of a wave with amplitude, vanishing at infinity, it is evident that the theory of Boussinesq leads to the same results as that of Korteweg and De Vries, if capillarity is neglected. A necessary and sufficient condition for the existence of a steady wave is a constant uniform wave velocity in all points of the wave surface.

6.1 The Solitary Steady Wave in the Theory of Boussinesq

It follows from (4.8) that the wave is stationary if $\omega = \sqrt{gH} + \frac{1}{2}\sqrt{g/H}h_1$, where h_1 is some as yet unknown constant, independent of x and t. Therefore,

$$\frac{\partial^2 h}{\partial x^2} = \frac{3h}{2H^3}(2h_1 - 3h) \tag{6.1}$$

Integration with $h \to 0$ and $\frac{\partial h}{\partial x} \to 0$ for $x \to \pm\infty$ gives

$$h(x,t) = h_1 \mathrm{sech}^2\left\{\sqrt{\frac{3h_1}{4H^3}}(x - \omega t)\right\}. \tag{6.2}$$

It follows $h_1 \geq h(x,t)$ and h_1 is the crest of the wave. The wave velocity

$$\omega = \sqrt{gH} + \frac{1}{2}\sqrt{\frac{g}{H}}h_1 \tag{6.3}$$

contains a correction of Lagrange's result with $\omega = \sqrt{gH}$ and was already experimentally verified in 1844 by Scott Russell [6]. The expressions (6.2) and (6.3) were also obtained by Rayleigh, see (3.6) and (3.7). It appears that the wave velocity is the larger the higher the crest of the wave and this means that in the case of several separate solitary waves of the form (6.2) the higher waves will overtake the lower ones whenever the higher waves were initially behind the lower waves. It is known that this occurs without change of form, however, there is only a phase shift.

The solitary waves behave like a row of rolling marbles, where the faster marbles carry over their impuls to the slower marbles. They were coined by Zabusky and Kruskal [3] as "solitons" to indicate their particle like properties. For an explicit calculation of this behaviour the interested reader may consult ref. [13], part II, 3.5.

6.2 The Solitary Steady Wave in the Theory of Korteweg and De Vries

Korteweg and De Vries do not have at their disposal an explicit expression for the wave velocity. However, for a steady wave in the moving (ξ, τ) frame (5.7) one has $\frac{\partial h}{\partial \tau} = 0$ and so by (5.8)

$$\frac{d}{d\xi}\left(\frac{1}{2}h^2 + \frac{2}{3}\alpha h + \frac{1}{3}\sigma\frac{d^2 h}{d\xi^2}\right) = 0 \tag{6.4}$$

where α is the still unknown correction of the wave velocity. Integration under the assumption h, $\frac{dh}{d\xi}$, $\frac{d^2h}{d\xi^2} \to 0$ for $\xi \to \pm\infty$ results into

$$\frac{dh}{d\xi} = \pm\sqrt{\frac{-h^2(h+2\alpha)}{\sigma}}.$$

There are two distinct cases, $\sigma > 0$ and $\sigma < 0$; we restrict our calculation to the case $\sigma > 0$ since the other case can be treated similarly. If $\sigma > 0$ then the constant 2α is negative and taking $2\alpha = -h_2$ one gets

$$h(\xi) = h_2 \mathrm{sech}^2\left(\sqrt{\frac{h_2}{4\sigma}}\xi\right), \tag{6.5}$$

with $h_2 > 0$ the crest of the wave. When the surface tension T is neglected, the parameter σ reduces to $\sigma = \frac{1}{3}H^3$ and (6.5) is in agreement with Boussinesq's result (6.2). Also the wave velocity is conform (6.3) because we have by (5.7)

$$\omega = \sqrt{gH} - \sqrt{\frac{g}{H}}\alpha = \sqrt{gH} + \frac{1}{2}\sqrt{\frac{g}{H}}h_2 \tag{6.6}$$

Korteweg and De Vries consider also solitary steady waves with a negative amplitude, possible for $\sigma < 0$, i.e. $H < \sqrt{\frac{3T}{\rho g}}$; this quantity equals approximately $\frac{1}{2}$ cm for water.

7 Periodic Stationary Waves

7.1 Boussinesq's Theory

Boussinesq investigates in his Mémoire *"Essai sur la theorie des eaux courrantes"* la *Forme la plus générale des intumescences propagées le long d'un canal horizontal et rectangulaire, qui avancent sans se déformer"* ([10], pp 390-396). By now, he does not have a formula for the wave velocity as in (4.8), because the condition $h(x,t) \to 0$ for $x \to \pm\infty$ does no longer hold in the case of periodic waves and so an easy evaluation of the wave form by setting ω constant is no longer possible. He introduces the mean horizontal velocity

$$U(x,t) = \frac{1}{H + h(x,t)} \int_0^{H+h(x,t)} u \, dy \tag{7.1}$$

which satisfies the relation

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x}\left(gh + \frac{1}{2}U^2 + \frac{H}{3}\frac{\partial^2 h}{\partial t^2}\right) = 0$$

see [10], 276, pp 390-391, or the paper by St. Venant [19] where a rather short derivation is presented.

For a steady wave we have $\frac{\partial h}{\partial t} = -\omega \frac{\partial h}{\partial x}$, $\frac{\partial^2 h}{\partial t^2} = \omega^2 \frac{\partial^2 h}{\partial x^2}$ and $\frac{\partial U}{\partial t} = -\omega \frac{\partial U}{\partial x}$, so the latter equation becomes

$$-\omega U + g(h - H) + \frac{U^2}{2} + \frac{H\omega^2}{3}\frac{\partial^2 h}{\partial x^2} = constant := \frac{\omega^2}{2H^2}c', \tag{7.2}$$

with ω the unknown constant wave velocity and $\frac{\omega^2}{2H^2}c'$ the constant of integration.

The mean velocity U is eliminated with the aid of the conservation of flux in a reference system bound to the wave, i.e.

$$\int_0^{H+h(x,t)} u dy = \omega h(x,t)$$

or

$$U = \frac{\omega h}{H + h} \approx \frac{\omega h}{H}\left(1 - \frac{h}{H}\right). \tag{7.3}$$

Substitution into (7.2) and omitting terms of order h^3 and higher we get

$$2\frac{\partial^2 h}{\partial x^2} = -\frac{3}{H^3}\{3h^2 - 2Hh\left(1 - \frac{gH}{\omega^2}\right) - c'\},$$

and by multiplication with $\frac{\partial h}{\partial x}$ and integration with respect to x

$$\left(\frac{\partial h}{\partial x}\right)^2 = -\frac{3}{H^3}\{h^3 - Hh^2\left(1 - \frac{gH}{\omega^2}\right) - c'h - c''\}, \tag{7.4}$$

with c'' the integration constant.

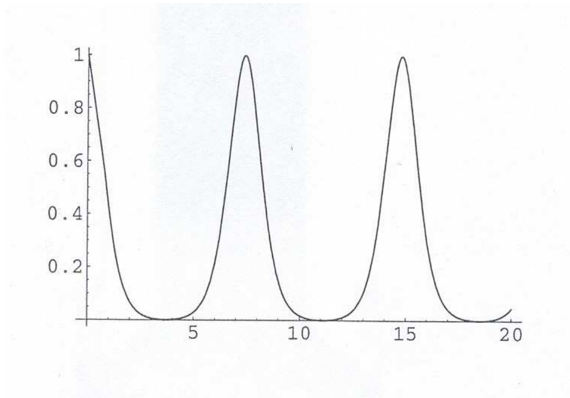

Figure 3: Cnoidal Wave

Suppose now that the minimum value h_0 of h, which is necessarily negative, is assumed at $x = 0$ and the wave surface represented by $\tilde{h}(x,t) = h(x,t) - h_0$, then $\tilde{h} \geq 0$ and $\tilde{h} = 0$ at x = 0. This shift results into

$$\left(\frac{\partial \tilde{h}}{\partial x}\right)^2 = -\frac{3}{H^3}\{\tilde{h}^3 + a_2\tilde{h}^2 + a_1\tilde{h} + a_0\}, \tag{7.5}$$

where a_0, a_1 and a_2 are certain constants, which due to the condition $\tilde{h} = 0$ and $\frac{\partial \tilde{h}}{\partial x} = 0$ at $x = 0$ satisfy $a_0 = 0$ and $a_1 < 0$. Therefore, (7.5) may be written as

$$\left(\frac{\partial \tilde{h}}{\partial x}\right)^2 = \frac{3}{H^3}\{(\tilde{h} + k)\tilde{h}(l - \tilde{h})\}, \tag{7.6}$$

with k and l positive constants.

Integration leads to a Jacobian elliptic function, but Boussinesq recommends to use Newton's binomium for a series expansion of the variable x into powers of $\tilde{h}$.

It follows from (7.6) that the solitary steady wave is only a particular case, resulting from (7.6) for $k \to 0$. Finally, one may substitute $h = \tilde{h} + h_0$ in (7.4), compare the coefficients of $\tilde{h}^2$ in (7.4) and (7.6) and one obtains after a simple evaluation the following expression for the wave velocity

$$\omega^2 = g\{H + (l - k)\}. \tag{7.7}$$

7.2 The Theory of Korteweg and De Vries

Since the Korteweg-de Vries equation (5.8), as derived in section 5, may also be applied to waves, not necessarily vanishing for $x \to \pm\infty$, the equation for the amplitude $h(\xi)$ of a steady wave is given by

$$\frac{d}{d\xi}\left(\frac{1}{2}h^2 + \frac{2}{3}\alpha h + \frac{1}{3}\sigma\frac{d^2h}{d\xi^2}\right) = 0. \tag{7.8}$$

Integrating this expression two times one obtains

$$c_1 + \frac{1}{2}h^2 + \frac{2}{3}\alpha h + \frac{1}{3}\sigma \frac{d^2 h}{d\xi^2} = 0$$

and

$$c_2 + 6c_1 h + h^3 + 2\alpha h^2 + \sigma \left(\frac{dh}{d\xi}\right)^2 = 0, \tag{7.9}$$

with c_1 and c_2 the constants of integration.

The wave surface may be defined as $y = H_0 + \tilde{h}(\xi)$ with H_0 the minimum value of y, $\tilde{h}(\xi) \geq 0$ and $\tilde{h}(0) = 0$.

It follows that $\frac{d\tilde{h}}{d\xi} = 0$ and $\frac{d^2\tilde{h}}{d\xi^2} > 0$ for $\tilde{h} = 0$ and so $c_2 = 0$ and $c_1 < 0$ under the assumption $\sigma > 0$. Consequently, the equation $\mu^2 + 2\alpha\mu + 6c_1 = 0$ has a positive root l and a negative root $-k$ and (7.9) reads

$$\frac{d\tilde{h}}{d\xi} = \pm\sqrt{\frac{1}{\sigma}(\tilde{h} + k)\tilde{h}(l - \tilde{h})}, \tag{7.10}$$

which for $T = 0$ is the same as (7.6).

With the aid of the substitution $\tilde{h} = l\cos^2\chi$ Korteweg and De Vries obtain the well-known periodical “cnoidal” wave

$$\tilde{h}(\xi) = l\,\mathrm{cn}^2\left(\sqrt{\frac{l+k}{4\sigma}}\xi\right), \tag{7.11}$$

where cn denotes one of the Jacobian elliptic functions with modulus $M = \frac{l}{l+k}$, period

$$4K = 4\int_0^1 (1 - t^2)^{-\frac{1}{2}}(1 - M^2 t^2)^{-\frac{1}{2}}\,dt$$

and wave length $4K\sqrt{\frac{\sigma}{l+k}}$.

This wave length becomes infinitely large for $k \to 0$ and the result is the solitary steady wave (6.5). However, one gets for large values of k, i.e. for small values of M, the sinusoidal wave

$$\tilde{h}(\xi) = l\cos^2\chi = l\cos^2\left(\sqrt{\frac{l+k}{4\sigma}}\xi\right)$$

with decreasing wave length for increasing k. This agrees with a result of Stokes [24] and in this case $\tilde{h}(\xi)$ may be expanded in a Fourier series; this may be the reason why Stokes at first believed that the only permanent wave should be of sinusoidal type.

The approach of Korteweg and De Vries as given here is in particular attractive, because of the central role of their equation (5.8) to which they frequently revert in the development of their theory.

8 The Stability of the Stationary Solitary Wave

It follows from the (x, t) dependence of the wave velocity ω, (4.8), that wave propagation involves in general a change of form, but by definition this does not occur in the case of a steady wave and so the question arises why the steady solitary wave is stable and an exception to the rule. For the possible existence of the steady wave a further investigation is required, in particular with regard to the "parameters" determining the stable behaviour. This has been carried out by Boussinesq and Korteweg-de Vries in quite different ways. The presence of the non-linear term $h\frac{\partial h}{\partial x}$ and the dispersion term $\frac{1}{9}H^3\frac{\partial h^3}{\partial x^3}$ is already an indication for a possible balance, furthering the stability of the wave.

8.1 Stability in the Theory of Korteweg-de Vries

The authors consider a wave form close to that of the steady solitary wave

$$h(\xi) = \bar{h}\,\mathrm{sech}^2(p\xi) \tag{8.1}$$

where $\bar{h}$ and p are as yet arbitrary constants with p near $\sqrt{\frac{\bar{h}}{4\sigma}}$ (see (6.5).

The deformation of this wave is determined by the equation (5.8) and substitution of (8.1) gives an equation for the evolution of the surface of the wave, given by $y = h(\xi, \tau)$:

$$\frac{\partial h}{\partial \tau} = 3\sqrt{\frac{g}{H}}\bar{h}p(4\sigma p^2 - \bar{h})\left\{-\mathrm{sech}^2(p\xi) + \frac{2}{3}\frac{\alpha + 2\sigma p^2}{4\sigma p^2 - \bar{h}}\right\}\mathrm{sech}^2(p\xi)\tanh(p\xi). \tag{8.2}$$

Taking $\alpha = 4\sigma p^2 - \frac{3}{2}\bar{h}$ this equation becomes

$$\frac{\partial h}{\partial \tau} = 3\sqrt{\frac{g}{H}}\bar{h}p(4\sigma p^2 - \bar{h})\mathrm{sech}^2(p\xi)\tanh^3(p\xi). \tag{8.3}$$

The choice $p = \sqrt{\frac{\bar{h}}{4\sigma}}$ and thus $\alpha = -\frac{1}{2}\bar{h}$ results into $\frac{\partial h}{\partial \tau} = 0$ and we get the steady wave (6.5).

A numerical analysis of (8.3) shows that the wave in its course becomes steeper in front and less steep behind when $p < \sqrt{\frac{\bar{h}}{4\sigma}}$ and conversely when $p > \sqrt{\frac{\bar{h}}{4\sigma}}$.

This result is in contradiction with the assertion of among others Airy, that a progressive wave always gets steeper in front and less steep behind. This opinion is conceivable if the dispersion is neglected.

8.2 Stability in the Theory of Boussinesq

Boussinesq considers waves, not necessarily steady, with the same energy

$$\rho g E = \frac{1}{2}\rho g\int_{-\infty}^{\infty} h^2 dx + \frac{\rho}{2}\int_{-\infty}^{\infty} dx\int_0^{H+h}(u^2 + v^2)dy = \rho g\int_{-\infty}^{\infty} h^2 dx \tag{8.4}$$

see ([9], pp 85-86).

Furthermore, he introduces the functional

$$M = \int_{-\infty}^{\infty} \left\{ \left(\frac{\partial h}{\partial x} \right)^2 - \frac{3}{H^3} h^3 \right\} dx. \tag{8.5}$$

which he calls the "moment de stabilité" and he shows that M is a conserved quantity, i.e. independent on t, ([9], pp 87-88, 97-99).

After the transformation

$$\varepsilon = \int_{x}^{\infty} h^2 dx$$

the expression for M becomes

$$M = \int_{0}^{E} \left\{ \left(\frac{1}{4} \frac{\partial h^2}{\partial \varepsilon} \right)^2 - 3 \frac{h}{H^3} \right\} d\varepsilon. \tag{8.6}$$

Boussinesq uses, without reference to Euler-Lagrange, the well-known method to obtain a condition for $h(\varepsilon, t)$ in order that M attains an extremal value and the result is

$$1 + \frac{2H^3}{3} h \frac{\partial}{\partial \varepsilon} \left(h \frac{\partial h}{\partial \varepsilon} \right) = 0 \tag{8.7}$$

From $d\varepsilon = -h^2 dx = h d\sigma$ follows equation (4.10) with $\frac{dh}{dt} = 0$ and therefore only the stationary solitary wave with given energy E yields an extremum for M. Variation of h with Δh gives $M > 0$ for all $h(\varepsilon, t)$ and so the extremum of M is a minimum. The stability of the wave is evident, because also ΔM does not depend on t.

As in discrete mechanical systems conserved quantities are of fundamental importance, also in continuous dynamical systems. Besides the integral invariants $Q = \int_{-\infty}^{\infty} h \, dx$ and $E = \int_{-\infty}^{\infty} h^2 \, dx$, corresponding with the conservation of mass and energy, Boussinesq discovered a third invariant, the "moment de stabilité".

He also showed that the velocity of the centre of gravity of a wave does not depend on time and this implies a fourth invariant of the motion ([9], pp 83-84; [16], p 135).

Conserved functionals may be considered as Hamiltonians in continuous dynamical systems and they play there a role analogous to the Hamilton functions in discrete systems. These continuous dynamical systems have been investigated only rather recently; the first fundamental results have been established in the seventies by Lax [27], Gardner [28], Zacharov [29] and Broer [30]. Nowadays, there exists an extensive literature on this subject; a valuable introduction with many references is the textbook by P.J. Olver [31].

The Korteweg-de Vries equation is the prototype of an integrable system with an infinite number of conserved functionals, mutually in involution with

respect to a suitably defined Poisson bracket. In particular the Korteweg-de Vries equation may be represented as a Hamiltonian system in the form

$$\frac{\partial h}{\partial t} = -\sqrt{gH}\frac{\partial}{\partial x}\delta_h(\mathcal{H}) \tag{8.8}$$

with

$$\mathcal{H}(h) = \int_{-\infty}^{\infty}\left[\frac{1}{2}h^2 + \varepsilon\left\{\left(\frac{\partial h}{\partial x}\right)^2 - \frac{3}{H^3}h^3\right\}\right]dx;$$

$\delta_h\mathcal{H}$ is the variational derivative of the Hamilton functional $\mathcal{H}$ and ε a scale parameter, ref. [13 part I, ch2; part II, ch 5].

The first term in $\mathcal{H}$ is the Hamilton functional for waves in the Lagrange approximation and the second term is the Boussinesq correction, given by the "moment de stabilité" M.

Hamilton's theory for finite discrete systems dates from about 1835 and it was a century after Boussinesq that this theory has been generalized for continuous systems. Boussinesq has set, by using functionals, a first step into the direction of this generalization.

9 Concluding Remarks

We have discussed in the preceding sections the more important aspects of the work of Boussinesq and Korteweg-de Vries, who have besides these also studied other specific topics such as the velocity field, the path of the fluid particles and the motion of the centre of gravity of a solitary wave.

Boussinesq finishes his article in the Journal de Mathématiques Pures et Appliquées with a qualitative examination of the change of form of long non-stationary waves and an attempt to prove that a positive solitary wave can be splitted into several other solitary waves. Korteweg and De Vries wanted to show that their approximation of the surface of a steady wave may be improved indefinitely, resulting in a convergent series. They claim in the introduction of their paper that *"in a frictionless liquid there may exist absolutely stationary waves and that the form of their surface and the motion of the liquid below it may be expressed by means of rapidly convergent series"*. The calculations, however elementary, are so complicated and tedious that one may expect that these have not received much attention. Even the second approximation (p. 443), following on the formulae (6.5) and (7.11) of the present paper requires already so much effort that it is reasonable to be content with the first approximation as given in (6.5) and (7.11).

It is somewhat surprising that Korteweg and De Vries refer in their paper only to Boussinesq's short communication in the Comptes Rendus of 1871 [7] and not to the extensive article in the J. Math. Pures et Appl. [9] and the "Essai sur la théorie des eaux courantes" [10] in 1872, respectively 1877. However, we should realize that the international exchange of scientific achievements in those days was not at the level as it is today.

As to the credit of the "a priori demonstration a posteriori" of the stable solitary wave, this credit belongs, of course, to M. Boussinesq. On the other hand, Korteweg and De Vries merit to be acknowledged for removing doubts on the existence of the "Great Wave" and for their contribution to the theory of long waves in shallow water.

Acknowledgement

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